1. Interior regularity of classical solutions II

Let us define

$$\|u\|_{H^2(\Omega)} = \Big(\int_\Omega u^2 + |\nabla u|^2 + |\nabla^2 u|^2 dx \Big)^{\frac{1}{2}}. \tag{1}$$

Theorem 1. *Suppose that $u \in C^\infty(\Omega)$ solves $\Delta u = f$ for $f \in C^\infty(\Omega)$. Then, for each compact set $K \subset \Omega$, there exists some constant C depending on K, Ω, f such that*

$$\|u\|_{H^2(K)} \leqslant C\|f\|_{L^2(\Omega)}.$$

Proof. Let us choose a compact set $\bar{K} \subset \Omega$ such that $K \subset \operatorname{int}(\bar{K})$, namely $K \subset \bar{K} \subset \Omega$ and they does not share their boundaries. Then, we choose a cut-off function $\eta \in C_c^\infty(\bar{K})$ such that $\eta = 1$ on K. Let us define

$$v = -\nabla_1(\eta^2 \nabla_1 u).$$

Then, we have

$$\begin{aligned}
\int_{\bar{K}} \nabla u \cdot \nabla v dx &= -\int_{\bar{K}} \nabla_i u \nabla_1 \nabla_i (\eta^2 \nabla_1 u) dx \\
&= \int_{\bar{K}} (\nabla_1 \nabla_i u)\big[2\eta \nabla_i \eta \nabla_1 u + \eta^2 \nabla_i \nabla_1 u\big] dx \\
&\geqslant \int_{\bar{K}} -\frac{1}{2}\eta^2 |\nabla\nabla_1 u| - 2|\nabla \eta|^2 |\nabla_1 u|^2 + \eta^2 |\nabla\nabla_1 u|^2 dx \\
&\geqslant \frac{1}{2}\int_{\bar{K}} \eta^2 |\nabla\nabla_1 u|^2 dx - C\int_{\bar{K}} |\nabla u|^2 dx.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_{\bar{K}} \nabla u \cdot \nabla v dx = -\int_{\bar{K}} \Delta u v dx = -\int f v dx = \int_{\bar{K}} f\big[\eta^2 \nabla_1 \nabla_1 u + 2\eta \nabla_1 \eta \nabla_1 u\big] dx \\
\leqslant \int_{\bar{K}} f^2 \eta^2 + \frac{1}{4}\eta^2 |\nabla_1 \nabla_1|^2 + f^2 + \eta^2 |\nabla_1 \eta|^2 |\nabla_1 u|^2 dx.
\end{aligned}$$

Combining the inequalities above yields

$$\int_K |\nabla\nabla_1 u|^2 dx \leqslant \int_{\bar{K}} \eta^2 |\nabla\nabla_1 u|^2 dx \leqslant C\int_{\bar{K}} f^2 + |\nabla u|^2 dx.$$

Since we have $\int_{\bar{K}} |\nabla u|^2 dx \leqslant C\int_\Omega f^2 dx$ by the result in the previous lecture notes, we can conclude

$$\int_K |\nabla\nabla_1 u|^2 dx \leqslant C\int_{\bar{K}} f^2 dx.$$

This implies the desired result.

□

2. Hilbert space

A metric space (X, d) is a pair of a set X and a distance d, where $d : X \times X \to \mathbb{R}$ satisfies

$$d(x,y) \geqslant 0, \quad \text{and } d(x,y) = 0 \text{ iff } x = y.$$
$$d(x,y) = d(y,x),$$
$$d(x,y) \leqslant d(x,z) + d(y,z).$$

A normed vector space $(X, \|\cdot\|)$ is a vector space X equipped with a norm $\|\cdot\|$ satisfying

$$\|x\| \geqslant 0, \quad \text{and } \|x\| = 0 \text{ iff } x = 0.$$
$$\|\lambda x\| = |\lambda|\|x\|,$$
$$\|x+y\| \leqslant \|x\| + \|y\|,$$

where $\lambda \in \mathbb{R}$.

We observe that a normed vector space $(X, \|\cdot\|)$ is a metric space with the distance $d(x,y) = \|x-y\|$.

Definition 2. A Banach space is a complete normed vector space.

We recall that a complete metric space X is that a Cauchy sequence $\{x_m\}_{m=1}^{\infty} \subset X$, namely $d(x_i, x_j) \to 0$ as $i, j \to +\infty$, converges to a point $\bar{x} \in X$.

An inner product space $(X, \langle , \rangle)$ is a vector space X equipped with an inner product $\langle , \rangle : X \times X \to \mathbb{R}$ satisfying

$$\langle x, x \rangle \geqslant 0, \quad \text{and } \langle x, x \rangle = 0 \text{ iff } x = 0.$$
$$\langle x, y \rangle = \langle y, x \rangle,$$
$$\langle x, ay + bz \rangle = a\langle x, y \rangle + b\langle x, z \rangle,$$

where $a, b \in \mathbb{R}$. We observe that an inner product space $(X, \langle , \rangle)$ is a normed space with the norm $\|x\| = \sqrt{\langle x, x \rangle}$.

Definition 3. A Hilbert space is a complete inner product space.

3. Projection

Definition 4. Given an inner product space X, $x, y \in X$ is orthogonal or normal if $\langle x, y\rangle = 0$. Given a closed subspace $V \subset X$, we denote by $V^\perp$ the orthogonal complement

$$V^\perp = \{x \in X : \langle x, v\rangle = 0 \,\text{for all} v \in V\}.$$

Definition 5. We denote by $P_V x$ the projection of x into a closed subspace V such that

$$\|P_V x - x\| = \inf_{x \in V} \|v - x\|.$$

Theorem 6 (Theorem 6.12 in the textbook)**.** *Let V be a closed subspace of a Hilbert space H. Then, for all $x \in H$, there exists a unique projection $P_V x$. Moreover, $P_V x = x$ iff $x \in V$, and $Q_v x = x - P_V x \in V^\perp$ so that*

$$\|x\|^2 = \|P_V x\|^2 + \|Q_V x\|^2. \tag{2}$$

Definition 7. Suppose that a toplogical space X has a sequence $\{x_m\} \subset X$ such hat any non-empty open subset $U \subset X$ contains some x_i. Then, we call X is separable.

Proposition 8. *A separable space contains a dense sequence $\{x_m\} \subset X$.*

Proposition 9. *$L^2(\Omega)$ and $H^1_0(\Omega)$ are separable.*

We recall that the Stone-Weierstrass theorem shows that C^0 is separable. Since C^0 is dense in L^2, L^2 is also separable. Moreover, H^1_0 is a subset of L^2. Thus, H^1_0 is also separable.

Definition 10. An orthogonal basis in a separable Hilbert space H is a sequence $\{w_m\} \subset H$ such that $\langle w_i, w_j\rangle = \delta_{ij}$ and $x = \sum_{i=1}^\infty \langle x, w_i\rangle w_i$ for all $x \in H$.

By using the projection and the Gram-Schmidt process, we can obtain the following proposition.

Proposition 11 (Proposition 6.18 in the textbook)**.** *A separable Hilbert space H admits a countable orthogonal basis.*